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High-order perturbation expansion: application to a quantum thermodynamical system of fermions

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Abstract. The recently proposed method for investigating the large-order behaviour of the perturbative expansion is applied to a quantum thermodynamical system of interacting fermions. The investigation is performed in detail for a system in one space dimension, with the use of the Thomas–Fermi approximation in order to deal with many-fermion systems. Some observations about the influence of the number of dimensions on the large-order behaviour are presented.

1. Introduction

Renewed interest in an old problem of quantum field theory—the convergence properties of the perturbative series—is shown by a recent series of works devoted to the study of this problem (Lipatov 1977a, b, Brézin *et al* 1977a, b, Parisi 1977a, Brézin 1978). A particular application of the proposed methods to a thermodynamical system of interacting bosons in one space dimension was also recently proposed (Calucci *et al* 1979); an analogous fermionic system is investigated here. For the fermionic case a modification of the technique was developed (Parisi 1977b, Itzykson *et al* 1977, Balian *et al* 1978), and the analysis performed here follows those procedures. As foreseen in the above references, the final result is quite different from that which one obtains for a boson system: in one space dimension a convergent perturbative expansion is in fact found; in more than one dimension no definite result is obtained, but it appears clearly that the number of spatial dimensions strongly influences the result.

When dealing with fermions there is no point in looking for classical solutions of the interacting field, but it is still possible to find configurations having a role analogous to the classical solution of the Bose field; these configurations represent systems of many fermions in a large and quasi-constant external field. Systems of this kind are treated through the Thomas–Fermi approximation (Parisi 1977b, Itzykson *et al* 1977), so that a great simplification is obtained in dealing with the functional determinant arising from the integration over the fermionic variables.

The zero-dimensional case is trivial in the fermionic case†, and therefore one begins directly with the case of one space dimension.

† For the bosonic system the the zero-dimensional case helps in understanding some features of the general problem.

2. One space dimension

The starting point is the functional integral representation of the partition function (see e.g. Bernard 1974),

$$Z(\lambda^2) = \int e^{-\mathscr{A}}\, D\psi D\psi^+ \Big/ \int e^{-\mathscr{A}_0}\, D\psi D\psi^+, \tag{2.1}$$

with

$$\mathscr{A}_0 = \int_0^\beta d\tau \int dx(\psi^+ \partial_\tau \psi + \partial_x \psi^+ \partial_x \psi - \mu\psi^+\psi), \qquad \mathscr{A} = \mathscr{A}_0 - \lambda^2 \int_0^\beta d\tau \int dx(\psi^+\psi)^2.$$

The field ψ is a (two-component) Pauli spinor; μ is the chemical potential, which is positive in the present case.

With a standard device, i.e. through the introduction of the auxiliary real scalar field α, with no kinetic term, the integral is transformed into

$$Z = \int \exp\left(-\mathscr{A}_0 - \int_0^\beta \int d\tau\, dx(\lambda\psi^+\psi\alpha + \tfrac{1}{4}\alpha^2)\right)$$
$$\times D\psi D\psi^+ D\alpha \Big/ \int \exp\left(-\mathscr{A}_0 - \int_0^\beta \int d\tau\, dx\, \tfrac{1}{4}\alpha^2\right) D\psi D\psi^+ D\alpha, \tag{2.2}$$

which, in turn, after integration over the spinor fields gives the expression

$$Z = \int \det[\partial_\tau - \partial_x^2 - \mu - \lambda\alpha] \exp\left(-\int_0^\beta \int d\tau\, dx\, \tfrac{1}{4}\alpha^2\right) D\alpha \Big/ \int \det[\partial_\tau - \partial_x^2 - \mu]$$
$$\times \exp\left(-\int_0^\beta \int d\tau\, dx\, \tfrac{1}{4}\alpha^2\right) D\alpha. \tag{2.3}$$

Defining now the free Green function

$$(\partial_\tau - \partial_x^2 - \mu) G_0(\tau_1, \tau_2; x_1, x_2) = \delta(\tau_1 - \tau_2)\delta(x_1 - x_2)$$

and the complete Green function

$$(\partial_\tau - \partial_x^2 - \mu - \phi(x, \tau)) G_\phi(\tau_1, \tau_2; x_1, x_2) = \delta(\tau_1 - \tau_2)\delta(x_1 - x_2)$$

the known relation

$$\ln\det[\partial_\tau - \partial_x^2 - \mu - \lambda\alpha] - \ln\det[\partial_\tau - \partial_x^2 - \mu] = \mathrm{tr}\ln(1 - G_0\lambda\alpha) = -\mathrm{tr}\int_0^\lambda G_{\kappa\alpha}\alpha\, d\kappa \tag{2.4}$$

is obtained (Parisi 1977b), from which it appears very clearly that the main task is the evaluation of the Green function in the external field $\kappa\alpha$.

At this point the use of the Thomas–Fermi approximation enters in a fundamental way. The Green function $G_\phi(\tau, \tau; x, x)$ is approximated by the Green function in a constant external field having the value of the field ϕ at the point x and 'instant' τ,

$$G_\phi(\tau, \tau; x, x) = -\frac{1}{2\pi} \int dq \frac{1}{1 + e^{\beta(q^2 - \mu - \phi(x, \tau))}},$$

and correspondingly

$$-\mathrm{tr}\int_0^\lambda G_{\kappa\alpha}\alpha\,\mathrm{d}\kappa=\int_0^\beta \mathrm{d}\tau\int \mathrm{d}x\frac{1}{2\pi\beta}\int \mathrm{d}q[\ln(1+\mathrm{e}^{-\beta(q^2-\mu)+\beta\lambda\alpha(x,\tau)})-\ln(1+\mathrm{e}^{-\beta(q^2-\mu)})]. \tag{2.5}$$

The method of Brézin (1977a, b) consists of studying the behaviour of the functional integral in $\mathrm{D}\alpha$ for large values of $\lambda\alpha$. In this way one puts in evidence possible divergences of the integral that reflect themselves in divergent behaviour of the formal series $\Sigma_N\lambda^N Z_N$. If, however, one meets a convergent integral, giving rise to an entire function of λ, then the study of large λ (and then large $\lambda\alpha$) is equally useful for determining the behaviour of the series, since it corresponds to the study of the order and type of the entire function, and the determination of the rate of growth of the function, for $\lambda\to\infty$, also defines the behaviour of the coefficients Z_N for $N\to\infty$ (Boas 1954).

For very large $\lambda\alpha$ equation (2.5) is approximated by

$$-\mathrm{tr}\int_0^\lambda G_{\kappa\alpha}\alpha\,\mathrm{d}\kappa\sim\frac{2}{3\pi}(\lambda\alpha+\mu)^{3/2}+\mathrm{O}((\lambda\alpha+\mu)^{1/2})+\cdots,$$

so that the integral of equation (2.3) reduces to

$$\int \exp\left[\int_0^\beta\int \mathrm{d}\tau\,\mathrm{d}x\left(\frac{2}{3\pi}(\lambda\alpha+\mu)^{3/2}-\frac{1}{4}\alpha^2\right)\right]\mathrm{D}\alpha\Big/\int\exp\left(-\int_0^\beta\int \mathrm{d}\tau\,\mathrm{d}x\,\tfrac{1}{4}\alpha^2\right)\mathrm{D}\alpha. \tag{2.6}$$

It is evident that the approximation has been too drastic—the functional integral has been completely factorised in τ and x, because there is no longer any derivative in the integrand. The first correction that can be foreseen takes into account the non-uniformity in x of $\phi(\tau,x)$ (ϕ will be kept constant in τ throughout the calculation). Writing $\phi(\tau,x)=\phi(\tau,x_0)+(x-x_0)\phi'(\tau,x_0)+\ldots$ one can compute a perturbative correction to the Green function G_ϕ, that is now written as

$$\begin{aligned}G_\phi^{(2)}&=G_\phi(\tau,\tau;x_1,x_2)+\int G_\phi(\tau,\tau;x_1,x')(x'-x_0)G_\phi(\tau,\tau;x',x_2)\phi'(\tau,x_0)\,\mathrm{d}x'\\&\quad+\int\int G_\phi(\tau,\tau;x_1,x')(x'-x_0)G_\phi(\tau,\tau;x',x'')(x''-x_0)\\&\quad\times G_\phi(\tau,\tau;x'',x_2)(\phi'(\tau,x_0))^2\,\mathrm{d}x'\,\mathrm{d}x''.\end{aligned}$$

In the zeroth-order approximation

$$\begin{aligned}&G_\phi(\tau;x_1,x_2)\\&\quad=-\frac{1}{2\pi}\int \mathrm{d}q\frac{1}{1+\mathrm{e}^{\beta(q^2-\mu-\phi(\tau,x_0))}}\mathrm{e}^{\mathrm{i}q(x_1-x_2)}\\&\quad=G_\phi(\tau;x_1-x_0,x_2-x_0)\\&\quad=-\frac{1}{2\pi}\int \mathrm{d}q\,\Gamma(q)\,\mathrm{e}^{\mathrm{i}q(x_1-x_2)};\end{aligned}$$

putting $y = x - x_0$,

$$G_\phi^{(2)} = G_\phi(\tau; y_1, y_2) + \beta \int G_\phi(\tau; y_1, y')y' G_\phi(\tau; y', y_2)\phi' \, dy'$$

$$+\beta^2 \int\int G_\phi(\tau; y_1, y')y' G_\phi(\tau; y', y'')y'' G_\phi(\tau; y'', y_2)(\phi')^2 \, dy' \, dy''$$

$$= G_\phi(\tau; y_1, y_2) + \frac{\beta}{2\pi}\int\int dq_1 \, dq_2 \, \Gamma(q_1)(\mathrm{i}\,\partial_1\delta(q_1 - q_2))$$

$$\times \Gamma(q_2)\, e^{\mathrm{i}q_1 y_1 - \mathrm{i}q_2 y_2}\phi' \, dq_1 \, dq_2$$

$$-\frac{\beta^2}{2\pi}\int\int\int dq_1 \, dq_2 \, dq_3 \, \Gamma(q_1)(\mathrm{i}\,\partial_1\delta(q_1 - q_2))$$

$$\times \Gamma(q_2)(\mathrm{i}\,\partial_2\delta(q_2 - q_3))\Gamma(q_3)\, e^{\mathrm{i}q_1 y_1 - \mathrm{i}q_3 y_2}(\phi')^2.$$

Since finally the trace has to be calculated, $y_1 = y_2$, then Γ being a function of q^2, the first integral gives zero, and the second integral can be put in different forms, one of the simplest being

$$-\frac{\beta^2}{2\pi}(\phi')^2 \int dq \, \Gamma(q)(\partial_q \Gamma(q))^2 = -\frac{2}{\pi}\beta^4 \int q^2 \, dq \, \Gamma^3(q)(1 - \Gamma(q))^2(\phi')^2.$$

Now the integration $\int_0^\lambda G_{\kappa\alpha}^{(2)}\alpha \, d\kappa$ has to be performed. This amounts to calculating

$$-\frac{2}{\pi}\beta^4\left(\frac{\alpha'}{\alpha}\right)^2 \int_0^\lambda \kappa^2 \, d\kappa \int q^2 \, dq \, \Gamma^3(1-\Gamma^2)\alpha$$

$$= -\frac{2}{\pi}\beta^4\left(\frac{\alpha'}{\alpha}\right)^2 \int_0^{\lambda\alpha} \phi^2 \, d\phi \int q^2 \, dq \frac{e^{-\beta(q^2-\mu-\phi)/2}}{[2\cosh\frac{1}{2}\beta(q^2-\mu-\phi)]^5}$$

$$= -\frac{4}{\pi}\beta^4\left(\frac{\alpha'}{\alpha}\right)^2 \int_0^\infty q^2 \, dq \int_{q^2-\mu-\lambda\alpha}^{q^2-\mu} (\nu + \mu - q^2)^2 \frac{e^{\beta\nu/2}}{(2\cosh\frac{1}{2}\beta\nu)^5} \, d\nu.$$

The denominator of the integrand (in $\cosh^5$) says that the contribution to the whole integral comes mainly from the regions of very small $\beta\nu$. This fact suggests the substitution of the term $e^{\beta\nu/2}(2\cosh\frac{1}{2}\beta\nu)^{-5}$ simply by $\frac{1}{12}\delta(\beta\nu)$. In this way the integration over ν becomes trivial, and the subsequent integration over q becomes simple and yields, for large, positive $\lambda\alpha$,

$$-\frac{2\beta^3}{3\pi}\left(\frac{\alpha'}{\alpha}\right)^2 [\tfrac{1}{7}(\lambda\alpha + \mu)^{7/2} + \tfrac{2}{5}(\lambda\alpha + \mu)^{5/2} + \cdots],$$

so that

$$-\mathrm{tr}\int_0^\lambda G_{\kappa\alpha}\alpha \, d\kappa \sim \int_0^\beta d\tau \int dx \left[\frac{2}{9\pi}(\lambda\alpha)^{3/2} + \frac{2\beta^3}{21\pi}(\lambda\alpha)^{3/2}\left(\frac{\alpha'}{\alpha}\right)^2 + \ldots\right],$$

and the new expression for the integral of equation (2.6) is

$$\int \exp\left\{\int_0^\beta \int d\tau \, dx \left[\frac{2}{9\pi}\alpha^{3/2} + \frac{2\beta^3}{21\pi}\alpha^{3/2}\left(\frac{d\alpha}{dx}\right)^2 - \frac{\alpha^2}{4\lambda}\right]\right\} D\alpha \Big/ \int \exp\left(-\int_0^\beta \int d\tau \, dx \frac{\alpha^2}{4\lambda}\right) D\alpha, \tag{2.7}$$

where the substitution $\lambda\alpha \to \alpha$ has been used.

At this point one can go back to the method of projecting out the $2N$th coefficient of the perturbative series through the integral (Brézin *et al* 1977a)

$$Z_{2N}=\frac{1}{2\pi \mathrm{i}}\oint\frac{\mathrm{d}\lambda^2}{\lambda^2}\int \exp\left\{\int_0^\beta\int \mathrm{d}\tau\,\mathrm{d}x\left[\frac{2}{9\pi}\alpha^{3/2}+\frac{2\beta^3}{21\pi}\alpha^{3/2}\left(\frac{\mathrm{d}\alpha}{\mathrm{d}x}\right)^2-\frac{\alpha^2}{4\lambda}\right]\right.$$
$$\left.-N\ln\lambda^2\right\}\mathrm{D}\alpha\Big/\int\exp\left(-\int_0^\beta\int\mathrm{d}\tau\,\mathrm{d}x\frac{\alpha^2}{4\lambda}\right)\mathrm{D}\alpha. \qquad (2.8)$$

The saddle-point method for λ^2 gives $\lambda^2=\int_0^\beta\int\mathrm{d}\tau\,\mathrm{d}x\,\alpha^2/4N$, but the saddle point in α gives rise to an exceedingly complicated equation.

To obtain some definite result one can use a sort of variational method, choosing a parametrisation for the function α. A very simple choice is

$$\alpha=A\,\mathrm{e}^{-Bx^2}. \qquad (2.9)$$

With this parametrisation in the integral in equation (2.8), looking for a common saddle point in A, B, λ^2, the result is $\lambda^2\sim N^{1/5}$, $A\sim N^{2/5}$, $B\sim N^{-4/5}$; then the saddle-point contribution to Z_{2N} is

$$Z_{2N}\sim\mathrm{e}^{cN-(N\ln N)/5}\sim\mathrm{e}^{(c-1/5)N}(N!)^{-1/5}, \qquad (2.10)$$

with the real constant c calculable but not very interesting, because it depends on the parametrisation.

From this result the series is clearly convergent, and the Z_{2N} constant in sign. There are a number of observations that must be made at this point. The first concerns a check of the result (the behaviour, not the value of the constant c) with change of parametrisation. With parametrisations like

$$\alpha=A/(1+Bx^2)^{2n} \qquad (2.9')$$

the stability of the result has been checked; in particular the variation in n gives, for this parameter, an equation independent of N. Other kinds of parametrisation with simple piecewise differentiable functions were done, with the same result.

The second observation is that the result is consistent with the hypothesis that the main contribution to the integral originates from large and slowly varying α fields. In fact, the saddle point is such that A increases with increasing N while B decreases with increasing N, so that the field α becomes larger and smoother. This kind of result is also obtained with the other parametrisations.

Finally, it is clear that the computation is not really completed, because one should integrate over the oscillations, separate the zero modes, and so on. These operations are ill-defined when the true saddle point for α is not known; however, it is known that they do not modify the general property of convergence of the series, although for other reasons they are relevant, e.g. they eliminate the 'i' introduced in equation (2.8) by the Cauchy integral representation.

Since the result expressed in equation (2.10) is affected anyhow by many approximations, this important refinement will not be done, and this equation will be, for the moment at least, the final result.

3. More than one space dimension

The steps of the method can be repeated for more than one dimension in space without any new complications. For two dimensions, the final result, analogous to equation (2.7), assumes a simpler form; in fact it reads

$$\int \mathrm{D}\alpha \exp\left[\int_0^\beta \int \mathrm{d}\tau\, \mathrm{d}^2 r\left(\frac{1}{4}\alpha^2+\frac{1}{8}\beta^3\alpha^2(\nabla\alpha)^2-\frac{1}{4\lambda^2}\alpha^2\right)\right]\Big/\int \mathrm{D}\alpha \exp\left(-\int_0^\beta\int \mathrm{d}\tau\, \mathrm{d}^2 r \frac{\alpha^2}{4\lambda^2}\right), \tag{3.1}$$

so that the equation for the saddle point in α can be easily written as

$$\Delta\alpha^2=\frac{4}{\beta^3}\left(1-\frac{1}{\lambda^2}\right)\equiv K,$$

yielding:

$$\alpha^2=\tfrac{1}{4}Kr^2+f, \tag{3.2}$$

f being a harmonic function. However, putting solution (3.2) in equation (3.1) the exponent is certainly divergent, and therefore no useful saddle point is found†. It is possible to extend the formalism to an arbitrary number of dimensions. It is easy to write the action in D dimensions; then, with a parametrisation like that in equation (2.9), i.e. $\alpha = A\,\mathrm{e}^{-Br^2}$, one looks for a common saddle point in A, B, λ^2. For $D\neq 2$ the result is

$$\lambda^2\sim N^{(2-D)/(3D+2)}, \qquad A\sim N^{2/(3D+2)}, \qquad B\sim N^{-4/(3D+2)}.$$

The fact that the exponent of N in the expression for λ^2 changes sign when D crosses the value 2 shows the peculiar character of this number of dimensions; in fact, when the saddle-point value of λ^2 decreases with N a diverging series is found. Above 2 what is found is a decreasing but positive value of the saddle point in λ^2, in such a way that a divergent, non-Borel-summable series is generated.

4. Conclusions

In conclusion, the two main qualitative results that appear from this calculation are: the difference between the boson and the fermion case, shown by their different behaviour at $D=1$; the strong influence of the number of dimensions in the fermion case, as discussed in § 3.

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† With a parametrisation like that in equations (2.9) and (2.9′), no solution is found, which verifies that the approximation does not introduce fake solutions.